Steady thin discs

- Suppose changes in external conditions (such as mass transfer rate) are slower than $t_{visc} \sim R^2/\nu$.

Timescale for changes in radial structure of disc

Hence, a stable disc will settle to a steady state-state structure --> Set $\partial/\partial t=0$ and integrate mass conservation eqn. to get:

$$R\,\Sigma\,u_R = constant$$

– This represents the constant inflow of mass through each *point* in the disc, so:

Accretion rate

$$\dot{M} = 2\pi\,R\,\Sigma(-u_R)$$

($u_R < 0$).

- Angular momentum conservation gives:

torque; constant of integration, related to rate at which angular momentum flows on to star

$$R\Sigma u_R R^2\Omega = \frac{G}{2\pi} + \frac{C}{2\pi}$$

- Use full expression for torque G to get:

$$-\nu\Sigma\Omega' = \Sigma(-u_R)\Omega + \frac{C}{2\pi R^3}.$$

Inner boundary condition

- If star rotates slower than breakup, at equator:

$$\Omega_* < \Omega_K(R_*).$$

- If material is braked from Ω_K to Ω_* in a surface boundary layer, width ***b*** << ***R_****, then at point ***R*** = ***R_* + b*** where Ω'=0:

$$\Omega(R_* + b) = \left(\frac{GM}{R_*^3}\right)^{1/2}[1 + O(b/R_*)].$$

So constant of integration

$$C = 2\pi\,R_*^3\,\Sigma\,u_R\,\Omega(R_* + b)$$
$$= -\dot{M}\,(GMR_*)^{1/2}$$

This is the "spindown torque" on a slowly rotating central star.

Eliminating the viscosity

- Integrated angular momentum eqn becomes:

torque

$$2\pi\,R\,\Sigma\,u_R.\,R^2\,\Omega = G + C$$
$$\Rightarrow(-\dot{M}).(G\,M\,R)^{1/2} = -3\pi\,\nu\,\Sigma(G\,M\,R)^{1/2} - \dot{M}(G\,M\,R_*)^{1/2}$$

- Rearrange:

$$\nu\,\Sigma = \frac{\dot{M}}{3\pi}\left[1-\left(\frac{R_*}{R}\right)^{1/2}\right]$$

- Note that $\nu\Sigma$ appears in expression for viscous dissipation D(R) per unit disc face area.
- Use this expression to eliminate viscosity from D(R).

Luminosity radiated by disc

- Annulus of width dR radiates power:

disc has 2 sides

$$dL = 2.\,D(R).\,2\pi\,R.\,dR$$
$$= \frac{3\,G\,M\,\dot{M}}{2\,R^2}\left[1-\left(\frac{R_*}{R}\right)^{1/2}\right]dR$$

Integrate to get:

$$L_{disc} = 4\pi\int_{R_*}^{\infty} R.\,D(R).\,dR$$
$$= \frac{3\,G\,M\,\dot{M}}{2}\int_{R_*}^{\infty}\left[1-\left(\frac{R_*}{R}\right)^{1/2}\right]\frac{dR}{R^2}$$
$$= \frac{3\,G\,M\,\dot{M}}{2\,R_*}\int_0^1(1-y^{1/2})\,d\,y\,[\mathrm{subst.}\ y = R_*/R]$$
$$= \frac{G\,M\,\dot{M}}{2\,R_*}$$

i.e. half the total gravitational energy lost in falling from infinity to R_*.

Important timescales

- How long does it take for the disc to be depleted?
- How long does a convective protostar take to contract?
- How fast does a slowly spinning star gain angular momentum from the disc?

The viscous timescale

- Viscosity spreads initial ring in radius on typical timescale:

$$t_{\rm visc} \sim R^2 / \nu \sim R / u_R$$
$$\text{where } u_R \sim \nu / R$$

Radial drift speed

- Standard 'molecular' viscosity treats particle speed $\tilde{u} \sim c_s$, $\lambda \sim$ mean free path.
- Insufficient to explain accretion timescales seen in CVs.

Anomalous viscosity

- Eddy velocity $\tilde{u} < c_s$ (to prevent thermalization of turbulent motion by shocks)
- Characteristic eddy size $\lambda < H$ (can't have eddies bigger than disc thickness)
- Parametrized eddy viscosity:

$$\nu \sim \lambda \tilde{u} \equiv \alpha c_s H, \ \alpha \leq 1.$$

The famous Shakura-Sunyaev "α parameter"

$$c_s H = \frac{c_s^2}{u_\phi} R = \left(\frac{kT}{\mu m_{\rm H}}\right)\left(\frac{R^3}{GM}\right)^{1/2}$$

But, what is the local temperature?

Radial temperature distribution

Power radiated per unit disc face area:

$$D(R) = \frac{3 G M \dot{M}}{8\pi R^3}\left[1 - \left(\frac{R_*}{R}\right)^{1/2}\right] = \sigma T^4$$

Eff. temp. of optically thick disc

Eff. Temperature at large radii:

$$T^4 \simeq \frac{3 G M \dot{M}}{8\pi R^3 \sigma} \quad \text{for} \quad R \gg R_*$$

Define:

$$T_*^4 \equiv \frac{3 G M \dot{M}}{8\pi R_*^3 \sigma}, \text{ then } \left(\frac{T}{T_*}\right)^4 = \left(\frac{R_*}{R}\right)^3$$

$$\Rightarrow T = T_* \left(\frac{R}{R_*}\right)^{-3/4}$$

Energetics of the impact region

- Since $L_{disc} = 0.5\, L_{acc}$, other half of L_{acc} must be released very close to star
- If $R_{inner} = R_{star}$ for a slowly rotating star, then orbital kinetic energy must be dissipated in impact region.
- Where does it go?
  - Added to internal energy of star?
  - Or re-radiated locally?
- Observational evidence of re-radiation:
  - Featureless blue veiling continuum on optical spectrum.
  - Photometric evidence of hotspots at T~5000 to 8000K.

Disc lifetimes

- Observed TTS disc lifetimes ~ 2 to 3 Myr.
- Equate $t_{depletion} \sim t_{visc}$ for outer parts of disc where t_{visc} is longest:

$$t_{\rm visc} \sim \frac{R^2}{\alpha c_s H} = \frac{(GMR)^{1/2} \mu m_H}{\alpha k T(R)}, \text{ where}$$

$$T(R) = 9.5\left(\frac{R}{40\text{ AU}}\right)^{-3/4}\left(\frac{M}{1\, M_{\rm Sun}}\right)\left(\frac{\dot{M}}{10^{-7} M_{\rm Sun}\,{\rm y}^{-1}}\right) K$$

- At 40 AU, get

$$t_{\rm visc} \sim \frac{2.6\times 10^4}{\alpha}\,{\rm y}$$

Consistent with $\alpha \sim 10^{-2}$, as found for quiescent discs in CVs.