- **Total torque on inner ring by outer:**

$$G(R) = 2\pi R \nu \Sigma R^2 \Omega'$$

Boundary length

kinematic viscosity $\nu = \lambda \tilde{u}$

- **G(R) is negative for Keplerian rotation**

i.e. Inner rings lose angular momentum to the outer rings and the gas slowly spirals inwards

Keplerian rotation

- **Angular velocity at radius R:**

$$R[\Omega_K(R)]^2 = \frac{GM_*}{R^2}$$

$$\Rightarrow \Omega_K(R) = \left(\frac{GM_*}{R^3}\right)^{1/2}$$

- **So Ω decreases outward:**

$$\Rightarrow \Omega' = \frac{d\Omega}{dR} = -\frac{3GM_*}{2\Omega_K R^4}$$

- **Circular velocity:**

$$u_\phi(R) = R\Omega_K(R)$$

- **Radial drift speed:**

$$u_R < 0 \text{ near star}, \ |u_R| << u_\phi$$

Mass conservation

Mass of annulus, radius R, width ΔR

Mass flux out of annulus at radius R

$$\frac{\partial}{\partial t}(2\pi R.\Delta R.\Sigma) = u_R(R,t).2\pi R.\Sigma(R,t) - u_R(R+\Delta R,t) \times 2\pi(R+\Delta R).\Sigma(R+\Delta R,t)$$

Mass flux in to annulus at radius R+ΔR

Angular momentum conservation

Angular momentum of annulus, radius R, width ΔR

$$\frac{\partial}{\partial t}(2\pi R.\Delta R.\Sigma.R^2\Omega)$$

$$= u_R(R,t).2\pi R.\Sigma(R,t).R^2\Omega(R) - u_R(R+\Delta R,t).2\pi(R+\Delta R) \times \Sigma(R+\Delta R,t).(R+\Delta R)^2\Omega(R+\Delta R) + \frac{\partial G}{\partial R}\Delta R$$

Net flow of ang. mom. in at R and out at R+ΔR

Transport due to net effects of viscous torques

Limit ΔR-> 0

Mass conservation:

$$\boxed{R\frac{\partial \Sigma}{\partial t} + \frac{\partial}{\partial R}(R\Sigma u_R) = 0}$$

Angular momentum conservation:

$$\frac{1}{2\pi}\frac{\partial G}{\partial R} = R\frac{\partial}{\partial t}(\Sigma R^2\Omega) + \frac{\partial}{\partial R}(R\Sigma u_R R^2\Omega)$$

$$= R^2\Omega\cancel{\left[R\frac{\partial}{\partial t}(\Sigma) + \frac{\partial}{\partial R}(R\Sigma u_R)\right]}$$

0 from mass cons.

$$+ \Sigma.R\cancel{\frac{\partial}{\partial t}(R^2\Omega)}^{\,0} + R\Sigma u_R.\frac{\partial}{\partial R}(R^2\Omega)$$

$$\Rightarrow \boxed{R\Sigma u_R.\frac{\partial}{\partial R}(R^2\Omega) = \frac{1}{2\pi}\frac{\partial G}{\partial R}}$$

Time evolution of surface density

- **Combine mass and ang. mom. eqs, to eliminate u_R:**

$$R\frac{\partial \Sigma}{\partial t} = -\frac{\partial}{\partial R}(R\Sigma u_R) = -\frac{\partial}{\partial R}\left[\frac{1}{2\pi(R^2\Omega)'}\frac{\partial G}{\partial R}\right]$$

Prime denotes differentiation wrt R here.

- **Use:** $G(R,t) = 2\pi R\nu\Sigma R^2\Omega'$

- **and subst. for R.R²Ω' and (R²Ω)' using Keplerian expressions to get:**

$$\boxed{\frac{\partial \Sigma}{\partial t} = \frac{3}{R}\frac{\partial}{\partial R}\left[R^{1/2}\frac{\partial}{\partial R}(\nu\Sigma R^{1/2})\right]} \qquad \textbf{(1)}$$

Separation of variables

- **Assume ν = const & put $s = 2R^{1/2}$**

$$\frac{\partial}{\partial t}\left(R^{1/2}\Sigma\right)=\frac{12\nu}{s^2}\frac{\partial^2}{\partial s^2}\left(R^{1/2}\Sigma\right)$$

- **Write $R^{1/2}\Sigma = T(t)S(s)$:**

$$\frac{T'}{T}=\frac{12\nu}{s^2}\frac{S''}{S}=\text{constant}=-\lambda^2$$

Exponentials **Bessel functions**

Green function solution for Σ

- **Initial ring of mass m at $R=R_0$:**

$$\Sigma(R,t=0)=\frac{m}{2\pi R_0}\delta(R-R_0)$$

Dirac δ-function

- **Solution is:**

$$\Sigma(x,\tau)=\frac{m}{\pi R_0^2}\tau^{-1}x^{-1/4}\exp\left\{-\frac{(1+x^2)}{\tau}\right\}I_{1/4}(2x/\tau)$$

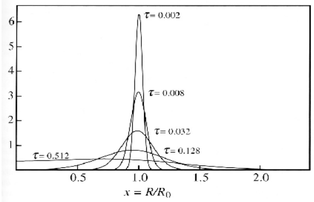
$$x=\frac{R}{R_0}\qquad \tau=\frac{12\nu t}{R_0^2}$$

Modified Bessel function

Viscous time scale

- **Viscosity has the effect of spreading an original ring out in radius on a typical timescale**

$$t_{visc}\sim R^2/\nu$$



- **Combining (1) and the mass continuity equation gives:**

$$u_R=-\frac{3}{\Sigma R^{1/2}}\frac{\partial}{\partial R}\left[\nu\Sigma R^{1/2}\right]$$

- **Hence** $u_R\sim\nu/R$ **and** $t_{visc}\sim R/u_R$

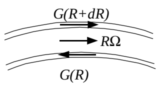
The role of viscosity in accretion discs

- **Transfers angular momentum along velocity gradients via random gas motions.**
- **Allows material to flow inward while transferring angular momentum outward.**
- **Causes dissipation within the gas, converting gravitational energy into heat and re-radiating it.**

Work and viscous torques

- **Torque on ring of gas between R and R+dR:**

$$G(R+dR)-G(R)=\frac{\partial G}{\partial R}dR.$$



- **Acts in same sense as Ω, so rate of working:**

$$\Omega\frac{\partial G}{\partial R}dR=\left[\frac{\partial}{\partial R}(G\Omega)-G\Omega'\right]dR$$

'Advection' of rotational energy through disc by torques, per unit radius dR.

Local rate of loss of mechanical energy to gas as internal (heat) energy via dissipative viscous torques, per unit radius dR.

Radiation of dissipated energy

- **Energy dissipated locally is radiated from disc faces at rate D(R) per unit area:**

$$D(R)=\frac{G\Omega'}{4\pi R}=\frac{1}{2}\nu\Sigma(R\Omega')^2$$

Each ring has 2 plane faces and hence area $4\pi R dR$

- **Note that $D(R)\geq 0$, vanishing only for rigid rotation.**

Disc thickness

- **No flow in z direction => hydrostatic equilibrium:**

$$\frac{1}{\rho}\frac{\partial P}{\partial z} = \frac{\partial}{\partial z}\left[\frac{GM}{\left(R^2+z^2\right)^{1/2}}\right] = -\frac{GMz}{R^3} \text{ for } z << R.$$

- **Define disc scale height H, with** $z \sim H$**:**

$$\frac{\partial P}{\partial z} \sim \frac{P}{H} \text{ and } P \sim \rho c_s^2 \Rightarrow \frac{H}{R} \cong c_s\left(\frac{R}{GM}\right)^{1/2} = \frac{c_s}{u_\phi}.$$

i.e. Keplerian orbital speed must be highly supersonic if H << R