Parker's isothermal wind solution

- Spherically symmetric
- Steadily expanding
- Isothermal

Mass continuity ← $4\pi r^2 \rho u = \text{const}$

Equation of motion ← $\rho u \frac{du}{dr} = -\frac{dp}{dr} - \rho \frac{GM_*}{r^2}$

Equation of state ← $p = \rho c_s^2$ where $c_s^2 = \frac{kT}{\mu m_H}$

The sonic point

- Eliminate ρ using:

$$\frac{dp}{dr} = c_s^2 \frac{d\rho}{dr} \text{ and } \frac{d}{dr}\left(r^2 \rho u\right) = 0$$

- To get: $\left(u - \frac{c_s^2}{u}\right)\frac{du}{dr} = \frac{2c_s^2}{r} - \frac{GM_*}{r^2}$

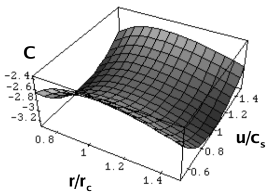
- Sonic point: both LHS and RHS vanish at critical point where: $u = c_s,\ r = r_c \equiv \frac{GM_*}{2c_s^2}$

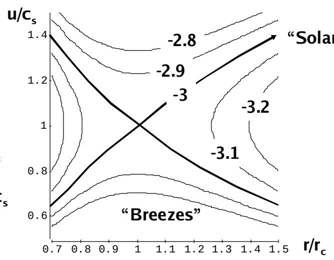
Isothermal wind solutions

- Integrate: $\left(\frac{u}{c_s}\right)^2 - \ln\left(\frac{u}{c_s}\right)^2 - 4\ln\left(\frac{r}{r_c}\right) - \frac{4r_c}{r} = C$

"Solar wind" solution passes smoothly through sonic point; putting $u=c_s$ and $r=r_c$ gives C = -3.

Constant of integration.

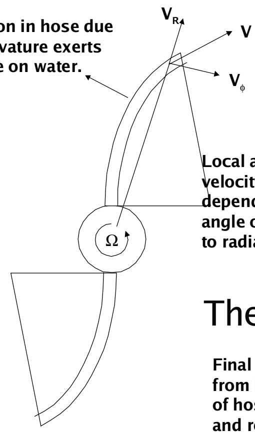




Rotation and magnetic field

- Field lines are anchored to Sun.
- Field lines must co-rotate at same angular velocity as Sun in steady state.
- How do wind flow and field-line shapes adjust themselves to achieve this?
- How does the angular momentum loss rate depend on:
  - wind density?
  - wind velocity?
  - stellar rotation rate?
  - magnetic field strength?

Tension in hose due to curvature exerts torque on water.



Water flows parallel to hose in corotating frame. Obeys mass continuity.

Hose bends backwards. This gives flow a backward ϕ-velocity to conserve angular momentum.

Local angular velocity of fluid depends on angle of hose to radial direction.

The sprinkler analogy

Final angular momentum emerging from hose depends on rigidity of hose material for a given flow rate and rotation rate.

Induction equation

- Wind flows along field:

Perfectly conducting → 0; Steady state → 0

$$\nabla \times (\boldsymbol{u} \times \boldsymbol{B}) + \eta \nabla^2 \boldsymbol{B} - \frac{\partial \boldsymbol{B}}{\partial t} = 0$$

$$\Rightarrow \boldsymbol{u} \times \boldsymbol{B} = \nabla \Phi$$ → Scalar electric potential

- Decompose vectors into 2 components:

$$\boldsymbol{u} = \boldsymbol{u}_p + u_\phi \boldsymbol{i}_\phi$$

$$\boldsymbol{B} = \boldsymbol{B}_p + B_\phi \boldsymbol{i}_\phi$$

Poloidal part in meridional plane incorporating cylindrical R and z components

Toroidal part in azimuthal (ϕ) direction

u_p and B_p are parallel

$$\boldsymbol{u}\times\boldsymbol{B} = \boldsymbol{u}_p\times\boldsymbol{B}_p + u_\phi \boldsymbol{i}_\phi \times \boldsymbol{B}_p + \boldsymbol{u}_p \times B_\phi \boldsymbol{i}_\phi + u_\phi \boldsymbol{i}_\phi \times B_\phi \boldsymbol{i}_\phi$$

Toroidal — Poloidal — Poloidal — Identically zero

- **System is axisymmetric, so:**

$$\boldsymbol{u}_p \times \boldsymbol{B}_p = 0$$
$$\Rightarrow \boldsymbol{u}_p = \kappa \boldsymbol{B}_p$$

Scalar function of position

Azimuthal velocity component

- **Expand everything in cylindricals to get:**

$$\nabla \times \left(u_\phi \boldsymbol{i}_\phi \times \boldsymbol{B}_p + \boldsymbol{u}_p \times B_\phi \boldsymbol{i}_\phi\right)$$
$$= \frac{\partial}{\partial R}\left(u_\phi B_R - \kappa B_R B_\phi\right) - \frac{\partial}{\partial z}\left(\kappa B_z B_\phi - u_\phi B_z\right)$$
$$= R\left(\nabla\cdot\boldsymbol{B}_p + \boldsymbol{B}_p\cdot\nabla\right)\left(\frac{u_\phi - \kappa B_\phi}{R}\right)$$
$$\Rightarrow \boldsymbol{B}\cdot\nabla\left(\frac{u_\phi - \kappa B_\phi}{R}\right) = 0$$

- **constant along field lines =>** $u_\phi - \kappa B_\phi = R\alpha(P)$

Local flow velocity

- **In the absence of a wind, a field line rotating at angular velocity $\alpha(P)$ has**

$$u_\phi = R\alpha(P) = R\Omega$$

isorotation — Uniform rotation

- **With a wind** $u_\phi = R\alpha(P) + \kappa B_\phi$
- **Uniform rotation** $u_\phi = R\Omega + \kappa B_\phi$
- **So** $u = \kappa B + R\Omega i_\phi$

Flow parallel to <u>total</u> field — Uniform rotation of field

Mass continuity

$$0 = \nabla\cdot(\rho\mathbf{u}) = \nabla\cdot(\rho\mathbf{u}_p) = \nabla\cdot(\rho\kappa\mathbf{B}_p) = \mathbf{B}_p\cdot\nabla(\rho\kappa)$$
$$\Rightarrow \rho\kappa = \rho\frac{u_p}{B_p} \equiv \psi = \text{constant along field-streamlines}$$

- **So both the material flux $\rho u_p A$ and the magnetic flux $B_p A$ along a flux tube stay constant as the tube cross-section A varies.**
- **For radial field and isotropic mass loss,**

$$\rho u_p = \frac{\dot{M}}{4\pi R^2} \text{ and } B_p = B_0\frac{R_*^2}{R^2}$$
$$\Rightarrow \psi = \frac{\dot{M}}{4\pi R_*^2 B_0}$$

Angular momentum conservation: 1

$$\rho(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = -\nabla p + \frac{1}{\mu_0}(\nabla\times\boldsymbol{B})\times\boldsymbol{B} - \frac{\rho GM}{r^2}\hat{\boldsymbol{r}}$$

- **Azimuthal component x R gives torque balance:**

Angular momentum transported by flow — Couple exerted by Lorentz force on unit volume

$$[\rho(\boldsymbol{u}\cdot\nabla)(R\boldsymbol{u})]_\phi = \left[\frac{R}{\mu_0}(\nabla\times\boldsymbol{B})\times\boldsymbol{B}\right]_\phi$$
$$\Rightarrow \rho\boldsymbol{u}\cdot\nabla\left(Ru_\phi\right) = \frac{R}{\mu_0}\left[-(\nabla\times\boldsymbol{B})_R B_z + (\nabla\times\boldsymbol{B})_z B_R\right]$$
$$= \frac{R}{\mu_0}\left[\frac{B_z}{R}\frac{\partial}{\partial z}\left(RB_\phi\right) + \frac{B_R}{R}\frac{\partial}{\partial R}\left(RB_\phi\right)\right]$$

Angular momentum conservation: 2

$$\nabla\left(Ru_\phi\right) \text{ has no } \phi\text{-component, so:}$$
$$\boldsymbol{u}\cdot\nabla\left(Ru_\phi\right) = \boldsymbol{u}_p\cdot\nabla\left(Ru_\phi\right) = \kappa\boldsymbol{B}_p\cdot\nabla\left(Ru_\phi\right)$$

- **Equation of motion becomes:**

$$\rho\kappa\boldsymbol{B}\cdot\nabla\left(Ru_\phi\right) = \boldsymbol{B}\cdot\nabla\left(\frac{RB_\phi}{\mu_0}\right)$$
$$\Rightarrow \boldsymbol{B}\cdot\nabla\left(\rho\kappa R u_\phi - \frac{RB_\phi}{\mu_0}\right) = 0.$$

constant

Net angular momentum flux

- **Integrate azimuthal equation of motion to get:**

$$\rho\kappa R u_\phi - \frac{R B_\phi}{\mu_0} \equiv \rho\kappa L = \text{constant}\,.$$

Constant of integration on each field-streamline. L is the net angular momentum per unit mass carried in the plasma motion and the magnetic stresses

- **Wind carries angular momentum away from star.**
- **Lorentz force transmits torque to stellar surface.**
- **Star spins down.**