## BACKGROUND: Maxwell's Equations (mks)

$$\underline{\nabla}.\underline{B}=0 \qquad \underline{\nabla}.\underline{E}=\rho/\epsilon$$

$$\underline{\nabla}\times\underline{E}=\frac{-\partial\underline{B}}{\partial t} \qquad \underline{\nabla}\times\underline{B}=\mu\,\underline{j}+\underbrace{\frac{1}{c^2}\frac{\partial\underline{E}}{\partial t}}$$

Can neglect if $v << c$

With $v_0=\frac{l_o}{t_o}$ ($v_0$: Typical plasma speed; $l_o$: Typical length scale; $t_o$: Typical timescale)

Since $\frac{E}{c^2 t}=\frac{E\,t}{c^2 t^2}\approx\frac{Bl}{c^2 t^2}=\frac{v^2}{c^2}\cdot\frac{B}{l}\approx\frac{v^2}{c^2}|\nabla\times B|$

We have used $\underline{H}=\underline{B}/\mu,\ \underline{D}=\epsilon\,\underline{E}$ to eliminate

**H** (the magnetic field) and **D** (the electric displacement)



## Definitions and Units

The magnetic permeability $\mu=\mu_0=4\pi\times10^{7}\,Hm^{-1}$

The permittivity of free space $\epsilon=\epsilon_0=8.854\times10^{12}\,Fm^{-1}$

$c^2=1/\mu_0\epsilon_0$

| | |
|---|---|
| Electric field | $E\,(Vm^{-1})$ |
| Charge density | $\rho\,(Cm^{-3})$ |
| Current density | $j\,(Am^{-2})$ |

Observers often measure magnetic induction in Gauss (rather than Tesla) and energy in erg (rather than Joules)

$$1\,G=10^{-4}\,T \qquad 1\,erg=10^{-7}\,J$$



## THE MAGNETOHYDRODYNAMIC (MHD) EQUATIONS: DERIVATION

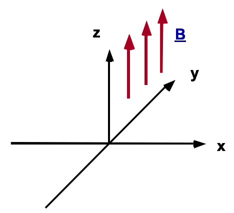
- These equations describe how a magnetic field interacts with a plasma (ionized gas).
- What happens when you put a magnetic field into a plasma?
- Force (F) on a particle of charge (q) moving in an electric field (E) and a magnetic field (B):

$$\underline{F}=q[\underline{E}(r,t)+\underline{\dot{r}}\times\underline{B}(r,t)]=m\,\underline{\ddot{r}}$$



## Simple case

- No electric field **E = 0**
- Uniform magnetic field **B = (0,0,B)**




Force equation $m\,\ddot{r}=q(\underline{\dot{r}}\times\underline{\hat{z}})\,B$

has components
$$m\,\ddot{x}=qB\,\dot{y}$$
$$m\,\ddot{y}=-qB\,\dot{x}$$



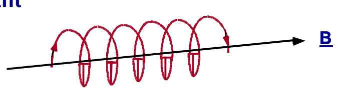
## Resultant particle motion

The solution describes helical motion

$$\dot{x}=v_\perp\cos(\Omega t+\alpha)$$
$$\dot{y}=-v_\perp\sin(\Omega t+\alpha)$$
$$\dot{z}=const.$$

where the gyro frequency (or Larmor frequency) $\Omega=\frac{qB}{m}$

and the perpendicular velocity is constant $v_\perp^2=\dot{x}^2+\dot{y}^2$




The gyroradius $r_L=\frac{v_\perp}{\Omega}$



## Consequences

- Charged particles are tied to fieldlines (electrons and protons gyrate in opposite directions).
- Even when the plasma is effectively collisionless ( mean free path >> typical lengthscales ) the magnetic field causes the plasma to behave collectively.
- The gyroradius defines the lengthscale on which particle motions are organised.

- Both conductivity and viscosity **across** magnetic field lines are much less than that **along** field lines.
- Applying a uniform electric field **perpendicular** to B causes the centre of the helix to drift in the direction of E, while applying an electric field **along** B causes particles to be accelerated.



## The Equation of Motion

- The equation of motion for charged particles is

$$n\,m\frac{D\underline{v}}{Dt}=n\,\underline{F}-\underline{\nabla}\,p+\underline{P}$$

where

**n** is the particle number density,
**m** is the particle mass,
**F** is a combination of the Lorentz force and gravity,

$$i.e.\ \underline{F}=q(\underline{E}+\underline{v}\times\underline{B})+mg$$

**p** is the pressure (assumed to be a scalar), and
**P** describes the momentum transfer by collisions.



Write this equation separately for electrons and ions:

$$n_e m_e\frac{D\underline{v}_e}{Dt}=-n_e q(\underline{E}+\underline{v}_e\times\underline{B})+n_e m_e\underline{g}-\underline{\nabla}\,p_e+\underline{P}_{ei}$$

$$n_i m_i\frac{D\underline{v}_i}{Dt}=Zn_i q(\underline{E}+\underline{v}_i\times\underline{B})+n_i m_i\underline{g}-\underline{\nabla}\,p_i+\underline{P}_{ie}$$

Add these using

$$Z\,n_i=n_e \qquad j=q(Z\,n_i\mathrm{v}_i-n_e\mathrm{v}_e)$$

$$\rho=n_i m_i+n_e m_e\approx n_i m_i \qquad \underline{v}=\frac{n_i m_i\underline{v}_i+n_e m_e\underline{v}_e}{n_i m_i+n_e m_e}$$



$$n_e m_e\frac{D\underline{v}_e}{Dt}+n_i m_i\frac{D\underline{v}_i}{Dt}=n_e q[(\mathrm{v}_i-\mathrm{v}_e)\times B]$$
$$-\nabla\,\underline{p}+(n_i m_i+n_e m_e)g$$

to get

$$\rho\frac{D\underline{v}}{Dt}=\underline{j}\times\underline{B}-\underline{\nabla}\,p+\rho\,g$$

This is just the same as the normal equation of motion for a fluid, except for the **Lorentz force** which describes the interaction between the magnetic field and the plasma.