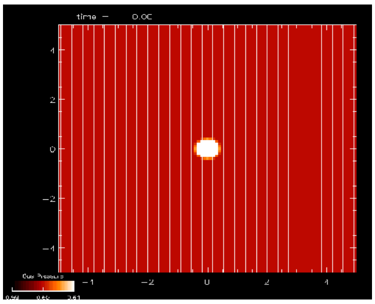
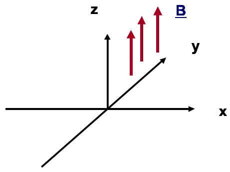
MHD Waves



MHD Waves

We want to examine the types of disturbances that can arise in a uniform medium with a uniform magnetic field in the z-direction.



We consider the effect of a small perturbation:

$$\underline{\boldsymbol{B}} = \underline{\boldsymbol{B}}_0 + \varepsilon \underline{\boldsymbol{B}}_1 + \varepsilon^2 \underline{\boldsymbol{B}}_2 + \ldots$$
$$\underline{\boldsymbol{v}} = \underline{\boldsymbol{v}}_0 + \varepsilon \underline{\boldsymbol{v}}_1 + \varepsilon^2 \underline{\boldsymbol{v}}_2 + \ldots$$

where ε is the perturbation parameter and $B_1 / B_0 = O(1)$

In the ambient medium we have no flow and a uniform field, pressure and density, i.e. to lowest order in ε

$$\underline{\boldsymbol{B}}_0 = B_0 \underline{\hat{\boldsymbol{z}}} \qquad \underline{\boldsymbol{v}}_0 = 0 \qquad p_0 = \text{constant} \qquad \rho_0 = \text{constant}$$

We expand each of the MHD equations as follows

$$\underline{\nabla} \cdot \underline{\boldsymbol{B}} = 0 \Rightarrow \underline{\nabla} \cdot \left(\underline{\boldsymbol{B}}_0 + \varepsilon \underline{\boldsymbol{B}}_1 + \varepsilon^2 \underline{\boldsymbol{B}}_2 + \ldots \right) = 0$$

and collect terms of the same order in ε

$$O(\varepsilon^0) \qquad \underline{\nabla} \cdot \underline{\boldsymbol{B}}_0 = 0$$
$$O(\varepsilon^1) \qquad \underline{\nabla} \cdot \underline{\boldsymbol{B}}_1 = 0$$

The full set of perturbed MHD equations to first order in ε is then, ignoring gravity and resistivity

$$\underline{\nabla} \cdot \underline{\boldsymbol{B}}_1 = 0 \tag{1}$$
$$\frac{\partial \rho_1}{\partial t} + \rho_0 \underline{\nabla} \cdot \underline{\boldsymbol{v}}_1 = 0 \tag{2}$$
$$\frac{\partial \underline{\boldsymbol{B}}_1}{\partial t} = (\underline{\boldsymbol{B}}_0 \cdot \underline{\nabla})\underline{\boldsymbol{v}}_1 - \underline{\boldsymbol{B}}_0 (\underline{\nabla} \cdot \underline{\boldsymbol{v}}_1) \tag{3}$$
$$\rho_0 \frac{\partial \underline{\boldsymbol{v}}_1}{\partial t} = -\underline{\nabla}\left(p_1 + \frac{\underline{\boldsymbol{B}}_0 \cdot \underline{\boldsymbol{B}}_1}{\mu} \right) + (\underline{\boldsymbol{B}}_0 \cdot \underline{\nabla}) \frac{\underline{\boldsymbol{B}}_1}{\mu} \tag{4}$$
$$\frac{\partial p_1}{\partial t} = c^2 \frac{\partial \rho_1}{\partial t} \tag{5}$$

The solutions to these equations tell us about the nature of the perturbation. We define the perturbed magnetic and total pressures as

$$p_{1m} = \frac{1}{\mu} \underline{\boldsymbol{B}}_0 \cdot \underline{\boldsymbol{B}}_1 \tag{6}$$
$$p_{1T} = p_1 + p_{1m} \tag{7}$$

and the perturbed velocities perpendicular to and parallel to the ambient field as

$$\underline{\boldsymbol{v}}_{\perp} = (v_{1x}, v_{1y}, 0)$$
$$\underline{\boldsymbol{v}}_{\parallel} = (0, 0, v_{1z})$$

Now from (3)

$$\frac{\partial \underline{\boldsymbol{B}}_1}{\partial t} = B_0 \frac{\partial \underline{\boldsymbol{v}}_1}{\partial z} - B_0 (\underline{\nabla} \cdot \underline{\boldsymbol{v}}_1) \underline{\hat{\boldsymbol{z}}} \tag{8}$$

and so from (6)

$$\frac{\partial p_{1m}}{\partial t} = \frac{1}{\mu} \underline{\mathbf{B}}_0 \cdot \frac{\partial \underline{\mathbf{B}}_1}{\partial t}$$
$$= \frac{B_0^2}{\mu} \left(\frac{\partial v_{1z}}{\partial z} - \frac{\partial v_{1x}}{\partial z} - \frac{\partial v_{1y}}{\partial z} - \frac{\partial v_{1z}}{\partial z} \right)$$
$$= -\frac{B_0^2}{\mu} \left(\underline{\nabla} \cdot \underline{\mathbf{v}}_{\perp} \right) \tag{9}$$

and from (5) and (2)

$$\frac{\partial p_1}{\partial t} = -c^2 \rho_0 \underline{\nabla} \cdot \underline{\mathbf{v}}_1 \qquad (10)$$

Hence,

$$\begin{aligned}\frac{\partial p_{1T}}{\partial t} &= -c^2 \rho_0 \underline{\nabla} \cdot \underline{\mathbf{v}}_1 - \frac{B_0^2}{\mu} \left(\underline{\nabla} \cdot \underline{\mathbf{v}}_\perp \right) \\ &= -c^2 \rho_0 \left(\underline{\nabla} \cdot \underline{\mathbf{v}}_\parallel + \underline{\nabla} \cdot \underline{\mathbf{v}}_\perp \right) - \frac{B_0^2}{\mu} \left(\underline{\nabla} \cdot \underline{\mathbf{v}}_\perp \right) \\ &= -\rho_0 \left(c^2 \frac{\partial v_\parallel}{\partial z} + \left(c^2 + v_A^2 \right) \left(\underline{\nabla} \cdot \underline{\mathbf{v}}_\perp \right) \right) \qquad (11)\end{aligned}$$

Taking $\partial / \partial t$ of the equation of motion (4) gives

$$\rho_0 \frac{\partial^2 \underline{\mathbf{v}}_1}{\partial t^2} = -\underline{\nabla} \left(\frac{\partial p_{1T}}{\partial t} \right) + \frac{1}{\mu} \left(\underline{\mathbf{B}}_0 \cdot \underline{\nabla} \right) \frac{\partial \underline{\mathbf{B}}_1}{\partial t}$$

and using (8) and (11) now gives

$$\begin{aligned}\rho_0 \frac{\partial^2 \underline{\mathbf{v}}_1}{\partial t^2} &= \underline{\nabla} \left(c^2 \frac{\partial v_\parallel}{\partial z} + \left(c^2 + v_A^2 \right) \left(\underline{\nabla} \cdot \underline{\mathbf{v}}_\perp \right) \right) \rho_0 \\ &\quad + \rho_0 v_A^2 \left(\frac{\partial^2 \underline{\mathbf{v}}_1}{\partial z^2} - \frac{\partial}{\partial z} \left(\underline{\nabla} \cdot \underline{\mathbf{v}}_1 \right) \hat{\underline{\mathbf{z}}} \right) \qquad (12)\end{aligned}$$

↑

Effects of tension

The x-, y-, and z-components of (12) give

$$\rho_0 \left(\frac{\partial^2}{\partial t^2} - v_A^2 \frac{\partial^2}{\partial z^2} \right) v_{1x} = -\frac{\partial}{\partial x} \left(\frac{\partial p_{1T}}{\partial t} \right) \qquad (13)$$

$$\rho_0 \left(\frac{\partial^2}{\partial t^2} - v_A^2 \frac{\partial^2}{\partial z^2} \right) v_{1y} = -\frac{\partial}{\partial y} \left(\frac{\partial p_{1T}}{\partial t} \right) \qquad (14)$$

$$\left(\frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial z^2} \right) v_{1z} = c^2 \frac{\partial}{\partial z} \left(\frac{\partial v_{1x}}{\partial x} + \frac{\partial v_{1y}}{\partial y} \right) \qquad (15)$$

We have reduced our set of perturbed MHD equations (1-5) with a single equation (12) for the perturbed velocity. Note that so far we have made no assumptions about the nature of the perturbation, other than that it is small.

We now look for plane-wave solutions to (13-15) of the form

$$v_{1x} = V_{1x} e^{i(\omega t - k_x x - k_y y - k_z z)}$$

for each of our unknowns v_{1x} v_{1y} v_{1z} p_{1T}

noting that, for example

$$\frac{\partial}{\partial x} \rightarrow -ik_x \qquad \frac{\partial}{\partial t} \rightarrow i\omega$$

We obtain for (13-15) and (11)

$$\rho_0 \left(\omega^2 - v_A^2 k_z^2 \right) V_{1x} = k_x \omega P_{1T} \qquad (16)$$

$$\rho_0 \left(\omega^2 - v_A^2 k_z^2 \right) V_{1y} = k_y \omega P_{1T} \qquad (17)$$

$$\left(\omega^2 - c^2 k_z^2 \right) V_{1z} = c^2 k_z \left(k_x V_{1x} + k_y V_{1y} \right) \qquad (18)$$

$$\omega P_{1T} = \rho_0 \left\{ c^2 k_z V_{1z} + \left(c^2 + v_A^2 \right) \left(k_x V_{1x} + k_y V_{1y} \right) \right\} \qquad (19)$$

These can be combined to give

$$\left(\omega^2 - v_A^2 k_z^2 \right) \left\{ \omega^4 - \omega^2 k^2 \left(c^2 + v_A^2 \right) + c^2 v_A^2 k^2 k_z^2 \right\} = 0 \qquad (20)$$

This dispersion relation $\omega(k)$ clearly has three solutions for ω^2. Before examining them in detail, we write down expressions for the forces that drive them.

From (10)

$$\omega P_1 = \rho_0 \left\{ c^2 \left(k_x V_{1x} + k_y V_{1y} \right) + c^2 k_z V_{1z} \right\} \qquad (21)$$

But, from (18)

$$\left(k_x V_{1x} + k_y V_{1y} \right) = \frac{\left(\omega^2 - c^2 k_z^2 \right) V_{1z}}{c^2 k_z} \qquad (22)$$

and so

$$P_1 = \rho_0 \frac{\omega}{k_z} V_{1z} \qquad (23)$$

Hence pressure perturbations are related to velocity perturbations along the magnetic field.

The magnetic pressure perturbation is from (9)

$$P_{1m} = \frac{\rho_0 v_A^2}{\omega}\left(k_x V_{1x} + k_y V_{1y}\right) \qquad (24)$$

or, using (22) and (23)

$$P_{1m} = \frac{\rho_0 v_A^2}{\omega}\frac{\left(\omega^2 - c^2 k_z^2\right)V_{1z}}{c^2 k_z} \qquad (25)$$

$$P_{1m} = \frac{\left(\omega^2 - c^2 k_z^2\right)}{\omega^2}\frac{v_A^2}{c^2}P_1 \qquad (26)$$

Finally, from (19)

$$\omega P_{1T} = \rho_0\left\{c^2 k_z V_{1z} + \left(c^2 + v_A^2\right)\left(k_x V_{1x} + k_y V_{1y}\right)\right\} \qquad (27)$$

or, using (22) and (23) and some algebra (!)

$$P_{1T} = \frac{P_1}{\omega^2 c^2}\left(c^2 + v_A^2\right)\left(\omega^2 - c_T^2 k_z^2\right) \qquad (28)$$

where $c_T^2 = \dfrac{c^2 v_A^2}{c^2 + v_A^2}$ is the tube or cusp speed

or, from (16) $$P_{1T} = \rho_0 \frac{\left(\omega^2 - v_A^2 k_z^2\right)}{\omega k_x}V_{1x} \qquad (29)$$

Lastly, from (12) if we write the magnetic tension as

$$\underline{\boldsymbol{f}}_1 = \left(\underline{\boldsymbol{B}}_0 \cdot \underline{\nabla}\right)\underline{\boldsymbol{B}}_1/\mu$$

then $$\frac{\partial \underline{\boldsymbol{f}}_1}{\partial t} = \rho_0 v_A^2\left(\frac{\partial^2 \underline{\boldsymbol{v}}_1}{\partial z^2} - \frac{\partial}{\partial z}\left(\underline{\nabla}\cdot\underline{\boldsymbol{v}}_1\right)\underline{\hat{\boldsymbol{z}}}\right) \qquad (30)$$

with components

$$\frac{\partial f_{1x}}{\partial t} = \rho_0 v_A^2 \frac{\partial^2 v_{1x}}{\partial z^2} \qquad (31)$$

$$\frac{\partial f_{1y}}{\partial t} = \rho_0 v_A^2 \frac{\partial^2 v_{1y}}{\partial z^2} \qquad (32)$$

$$\frac{\partial f_{1z}}{\partial t} = -\rho_0 v_A^2 \frac{\partial}{\partial z}\left(\frac{\partial v_{1x}}{\partial x} + \frac{\partial v_{1y}}{\partial y}\right) \qquad (33)$$

Fourier-analysing as $f_{1x} = F_{1x}e^{i(\omega t - \boldsymbol{k}\cdot\boldsymbol{r})}$ etc gives

$$F_{1x} = -\frac{\rho_0 v_A^2}{i\omega}k_z^2 V_{1x} \qquad (34)$$

$$F_{1y} = -\frac{\rho_0 v_A^2}{i\omega}k_z^2 V_{1y} \qquad (35)$$

$$F_{1z} = -\frac{\rho_0 v_A^2}{\omega}ik_z\left(k_x V_{1x} + k_y V_{1y}\right) \qquad (36)$$

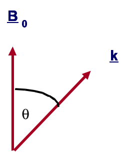
or, using (24) $$F_{1z} = -ik_z P_{1m} \qquad (37)$$

So, across the field, the perturbed tension is related to the perturbed velocity components, while along the field it is related to the perturbed magnetic pressure.

The three wave modes

Remember: phase speed ω/k

group velocity $\left(\dfrac{\partial\omega}{\partial k_x}, \dfrac{\partial\omega}{\partial k_y}, \dfrac{\partial\omega}{\partial k_z}\right)$



1. The Alfven wave

One solution of (20) is $\omega^2 - k_z^2 v_A^2 = 0$

i.e. $\omega^2 = k^2\cos^2\theta\, v_A^2$

phase speed $\omega/k = \pm v_A\cos\theta$

group velocity $(0, 0, \partial\omega/\partial k_z) = \pm v_A\underline{\hat{\boldsymbol{z}}}$

Nature of the perturbations:

From (29) $P_{1T} = 0$

and so from (28) $P_1 = 0$

and so from (26), (5) and (23)

$P_{1m} = 0 \qquad \rho_1 = 0 \qquad V_{1z} = 0$

Hence also from (37) $F_{1z} = 0$

Summary: the Alfven mode is an incompressible motion, transverse to the magnetic field and driven by tension forces. It is anisotropic, unable to propagate across the field since

$$\omega = 0 \quad \text{for} \quad \theta = \pi/2 \quad \text{and} \quad k_z = 0$$

2. Slow and fast waves

The other two solutions of (20) are the roots of

$$\omega^4 - \omega^2 k^2 \left(c^2 + v_A^2\right) + c^2 v_A^2 k^4 \cos^2\theta = 0$$

i.e.

$$\frac{2\omega^2}{k^2} = \left(c^2 + v_A^2\right) \pm \left\{\left(c^2 + v_A^2\right)^2 - 4c^2 v_A^2 \cos^2\theta\right\}^{1/2} \qquad (38)$$

Propagation: parallel to B_0 $\quad \theta \to 0 \Rightarrow \cos\theta \to 1$

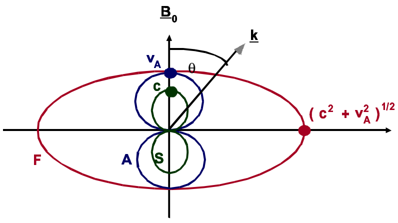
$$2\frac{\omega^2}{k^2} \to \left(c^2 + v_A^2\right) \pm \left(c^2 - v_A^2\right)$$

$$2\frac{\omega^2}{k^2} \to \begin{cases} \max\left(2c^2, 2v_A^2\right) & \text{fast wave} \\ \min\left(2c^2, 2v_A^2\right) & \text{slow wave} \end{cases}$$

perpendicular to B_0 $\quad \theta \to \pi/2 \Rightarrow \cos\theta \to 0$

$$2\frac{\omega^2}{k^2} \to \begin{cases} 2\left(c^2 + v_A^2\right) & \text{fast wave} \\ 0 & \text{slow wave} \end{cases}$$

We can illustrate the dependence of the phase speed on the angle of propagation with a polar diagram drawn for the case where $v_A > c$.



We note that only the fast wave propagates across the field and it has its greatest phase speed there.

Looking at the forces

Remember from (26) that the magnetic and plasma pressure perturbations are related:

$$\frac{P_{1m}}{P_1} = \frac{\left(\omega^2 - c^2 k_z^2\right)}{\omega^2}\frac{v_A^2}{c^2}$$

Depending on the phase speed of the wave, these two pressure perturbations may act together or in opposition:

$$P_{1m}/P_1 > 0 \quad \text{for} \quad \omega^2 > c^2 k_z^2 \qquad \text{(in phase)}$$

$$P_{1m}/P_1 < 0 \quad \text{for} \quad \omega^2 < c^2 k_z^2 \qquad \text{(out of phase)}$$

Slow wave: $\quad \omega^2 < c^2 k_z^2$

Hence the two pressure perturbations are always out of phase, so when one is trying to increase the local pressure, the other is acting to decrease it.

As $\theta \to \pi/2$ it can be shown that

$$P_{1m}/P_1 \to -1$$

and so the two pressure perturbations become exactly out of phase (so that the total pressure perturbation falls to zero). Since in this limit

$$k_z \to 0$$

the magnetic tension is also zero, (see 34-36) hence this mode cannot propagate across the field.

Fast wave: $\quad \omega^2 > c^2 k_z^2$

Hence the perturbations in magnetic and plasma pressures are always in phase. Clearly, P_{1m}/P_1 (and hence P_{1T}/P_1) has a maximum for

$$\theta = \pi/2 \quad \text{i.e.} \quad k_z = 0$$

The tension force again falls to zero as $\quad \theta \to \pi/2$

The fast wave has its maximum speed perpendicular to the field, when it is driven by plasma and magnetic pressure perturbations acting in phase.

The relative phases of P_{1m} and P_1 allow us to distinguish between the fast and slow modes.

Limiting cases:

1. Incompressible flow $\quad c^2 >> v_A^2$

The dispersion relation (38) reduces to

$$2\frac{\omega^2}{k^2} \approx c^2\left\{1 \pm \left(1 - 4\frac{v_A^2}{c^2}\cos^2\theta\right)^{1/2}\right\}$$

As $c^2 \to \infty$ this becomes simply

$$\frac{\omega^2}{k^2} \to \begin{cases} \infty & \text{fast wave} \\ v_A^2\cos^2\theta & \text{slow wave} \end{cases}$$

Hence in the incompressible limit, the fast wave disappears and the slow wave has the same dispersion relation as an Alfven wave:

$$\omega^2 \to k_z^2 v_A^2$$

However, its behaviour is quite different!!

For the slow wave: $\quad v_{1z} \neq 0 \quad P_1 \neq 0$

For the Alfven wave: $\quad v_{1z} = P_1 = 0$

2. Dominant magnetic field (low β) $\quad v_A^2 >> c^2$

The dispersion relation (38) becomes

$$2\frac{\omega^2}{k^2} \approx v_A^2\left\{1 \pm \left(1 - 4\frac{c^2}{v_A^2}\cos^2\theta\right)^{1/2}\right\}$$

and as $v_A^2 \to \infty$

$$\frac{\omega^2}{k^2} \to \begin{cases} v_A^2 & \text{fast wave} \\ c^2\cos^2\theta & \text{slow wave} \end{cases}$$

We can write this as

$$\omega^2 \to \begin{cases} k^2 v_A^2 & \text{fast wave} \\ k_z^2 c^2 & \text{slow wave} \end{cases}$$

Hence in the low-β limit, the slow wave propagates anisotropically, while the fast wave is isotropic.

The fast wave in this limit is sometimes (misleadingly) called the "compressional Alfven wave" but it is quite different from the Alfven wave.

Wave propagation

Low plasma β – i.e. strong magnetic field