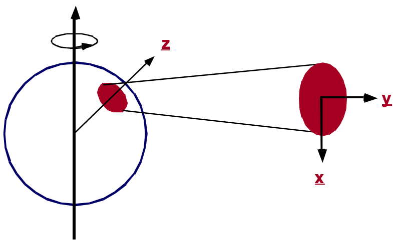
A simple dynamo

Consider a small element of surface on a thin, spherical shell.

z : outward normal
y : points east
x : points south



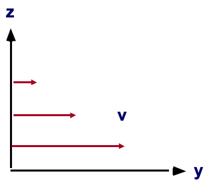
All quantities are uniform in y, i.e. $\frac{\partial}{\partial y}=0$

To ensure $\underline{\nabla}.B=0$ we write B in terms of a flux function

$$\underline{B}=\underline{\nabla}\times\underline{A}=\hat{\underline{x}}\left(\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}\right)-\hat{\underline{y}}\left(\frac{\partial A_z}{\partial x}-\frac{\partial A_x}{\partial z}\right)+\hat{\underline{z}}\left(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right)$$

giving $\underline{B}=\left(-\frac{\partial A_y}{\partial z},B_y,\frac{\partial A_y}{\partial x}\right)$

Assume a vertical shear in the flow. $\underline{v}=\mathrm{v}_y(z)$



This flow drags fieldlines along, so that

$$\underline{v}\times\underline{B}=\left(\mathrm{v}_y\frac{\partial A_y}{\partial x},0,\mathrm{v}_y\frac{\partial A_y}{\partial z}\right)$$

$$\underline{\nabla}\times(\underline{v}\times\underline{B})=\hat{\underline{y}}\left[-\frac{\partial}{\partial x}\left(\mathrm{v}_y\frac{\partial A_y}{\partial z}\right)+\frac{\partial}{\partial z}\left(\mathrm{v}_y\frac{\partial A_y}{\partial x}\right)\right]$$

$$=\hat{\underline{y}}\frac{d\,\mathrm{v}_y}{dz}\frac{\partial A_y}{\partial x}$$

Hence the y-component of the induction equation

$$\frac{\partial\underline{B}}{\partial t}=\underline{\nabla}\times(\underline{v}\times\underline{B})+\eta\nabla^2\underline{B}$$

gives

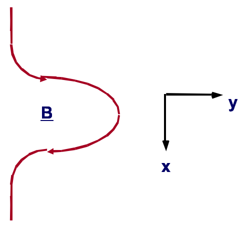
$$\left(\frac{\partial}{\partial t}-\eta\nabla^2\right)B_y=\frac{d\,\mathrm{v}_y}{dz}\frac{\partial A_y}{\partial x}$$

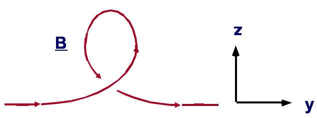
Since the advection term only has a y-component, the x- and z-components of the induction equation only give diffusive decay, e.g. for x:

$$\frac{\partial B_x}{\partial t}=\eta\nabla^2B_x$$

Hence the y-component of the field may increase, but the x- and z-components decay.

We want to mimic the effects of Coriolis forces which twist this flux tube as it rises, converting some of the y-component of the field into x- and z-components.





To do this we add an extra term to E

$$\underline{E}+\underline{E}_\alpha=-\underline{v}\times\underline{B}-\underline{j}/\sigma$$

where $\underline{E}_\alpha=\alpha B_y\hat{\underline{y}}$

The induction equation then becomes

$$\frac{\partial\underline{B}}{\partial t}=\underline{\nabla}\times(\underline{v}\times\underline{B}+\underline{E}_\alpha)+\eta\nabla^2\underline{B}$$

The y-component is unchanged, but we now have x- and z-components from the electric field we have added:

$$\frac{\partial\underline{B}}{\partial t}=\left(-\frac{\partial}{\partial z}(\alpha B_y),\frac{d\,\mathrm{v}_y}{dz}\frac{\partial A_y}{\partial x},\frac{\partial}{\partial x}(\alpha B_y)\right)+\eta\nabla^2\underline{B}$$

Writing the x- and z-components in terms of the flux function and integrating gives

$$\left(\frac{\partial}{\partial t}-\eta\nabla^2\right)A_y=\alpha B_y$$

where the constant of integration is zero by symmetry (since α must change sign at the equator). This is very similar to the y-component:

$$\left(\frac{\partial}{\partial t}-\eta\nabla^2\right)B_y=\frac{dv_y}{dz}\frac{\partial A_y}{\partial x}$$

We assume that $d\,v_y/dz$ is a constant and look for plane wave solutions

$$B_y=B_0 e^{[\omega t+i(k_x x+k_y y)]}$$
$$A_y=A_0 e^{[\omega t+i(k_x x+k_y y)]}$$

We note that

$$\left(\frac{\partial}{\partial t}-\eta\nabla^2\right)\rightarrow(\omega+\eta k^2)$$

where $k^2=k_x^2+k_y^2$

to give

$$(\omega+\eta k^2)A_0=\alpha B_0$$
$$(\omega+\eta k^2)B_0=v'_y ik_x A_0$$

With a little re-arranging this becomes

$$(\omega+\eta k^2)^2=v'_y ik_x\alpha$$

or

$$\omega=-\eta k^2\pm(ik_x\alpha v'_y)^{1/2}$$

Writing $K=k_x\alpha v'_y/2$

(where K may be positive or negative) and using

$$\sqrt{i}=(1+i)/\sqrt{(2)}$$

then gives

$$\omega=-\eta k^2\pm(1+i)K^{1/2}$$

We can separate ω into real and imaginary parts :

$$\omega=\underbrace{-\eta k^2\pm|K|^{1/2}}_{\mathrm{Re}}\pm i\underbrace{|K|^{1/2}}_{\mathrm{Im}(\omega)}$$

giving:

$$B_y=B_0\exp\{\mathrm{Re}(\omega)t\}\exp\{i[\mathrm{Im}(\omega)t+k_x x+k_y y]\}$$

The real part of ω gives a growing or decaying exponential, while the imaginary part gives oscillation. The behaviour of the magnetic field depends critically on the dynamo number:

$$N_D=\frac{K}{\eta^2 k^4}=\frac{\alpha k_x v'_y}{2\eta^2 k^4}$$

The real part

This may be positive (growing) or negative (decaying) depending on the magnitude of the dynamo number

$$|N_D|>1\rightarrow\mathrm{Re}(\omega)>0$$

The imaginary part:

The direction of travel of the dynamo waves depends on the sign of the dynamo number

$N_D<0$ waves travel southwards (positive $\hat{x}$)

$N_D>0$ waves travel northwards (negative $\hat{x}$)