Bohr-van Leeuwen Paradox

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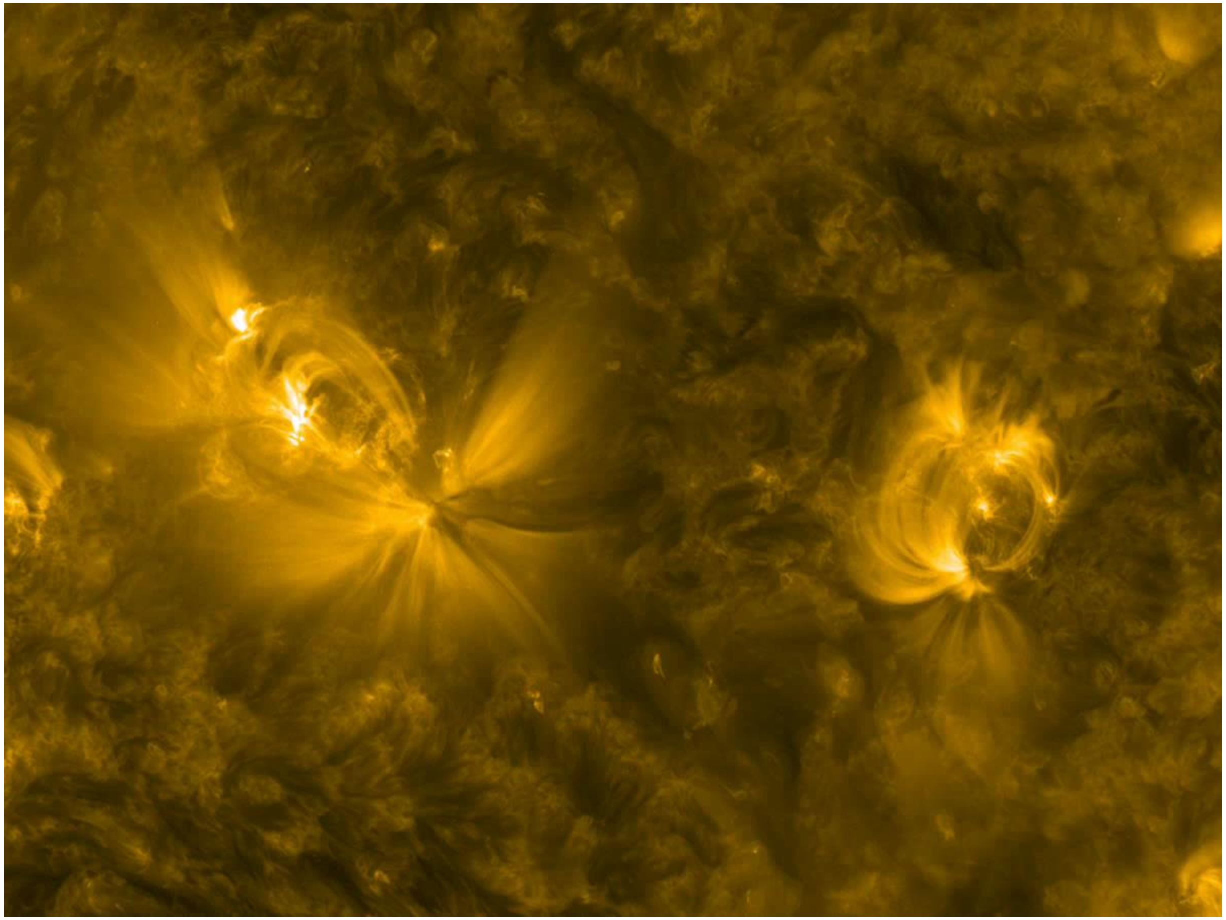
Outline

1. Plasma in solar active regions
2. Bohr-van Leeuwen paradox (Not theorem)
3. Influence of the paradox (classical physics, MHD)
4. Magnetic moment of thermal plasma
5. Kelvin force acting on thermal plasma
6. MHD equation of motion along magnetic field
7. Application to the solar atmosphere (steady upward flow, coronal heating, solar wind acceleration)

Plasma in solar active regions
SDO/AIA171 movie



Plasma in solar active regions
SDO/AIA171 movie

Q: What is the driving mechanism for continuous plasma flow from the central active region?

A: The Kelvin force acts on the magnetic moment of the thermal plasma (the plasma is a diamagnetic fluid and is therefore pushed upward where the magnetic field is weak).

- To prove this, the following items need to be derived.
    1. Derive a magnetic moment for thermal plasma. (contradict with Bohr-van Leeuwen theorem)
    2. Derive the Kelvin force for non-linear magnetic material from the Maxwell equations.
    3. Derive the Magnetohydrodynamic (MHD) equation of motion including the magnetic moment (Kelvin force, magnetization current).
- Apply the MHD equation of motion to the solar atmosphere and explain plasma upward flow along the magnetic field lines.

Bohr-van Leeuwen Paradox (Not Theorem)

- According to statistical thermodynamics, the physical quantity of a statistical ensemble at a constant temperature (T) is determined by the partition function (Z). In classical physics, the partition function depends on the Hamiltonian of the system. In a thermal plasma in a magnetic field, the Hamiltonian is the thermal kinetic energy of the particles (assuming no electric field). The magnetic field does not affect the kinetic energy (the Lorentz force is orthogonal to the velocity). Therefore, the magnetic field does not influence the physical quantity. Bohr [1911]
- However, the Hamiltonian of a particle in a magnetic field is a function of the generalized coordinates (r) and the generalized momentum ($p = mv + qA$) and depends on the magnetic field (B) via the vector potential (A). For a uniform magnetic field, $A = (1/2)B \times r$.
- van Leeuwen [1921] used the Hamiltonian (H) and the partition function with generalized coordinates and differentiated these by B to derive the magnetic moment $\mu = (q/2)\, r \times v$.
- Then, she attempted to evaluate this μ over the whole phase space (r, p), but this resulted in zero because she first integrated over p, ignoring the fact that r and p are linked via the equation of motion. If the equation of motion is solved and the relation between r and v is taken into account, the correct magnetic moment is obtained. Even if $\langle v \rangle = 0$ at each point in the plasma, $\langle r \times v \rangle \neq 0$.

Statistical thermodynamics

- Hamiltonian with cylindrical coordinates {r, ϕ, z} and the generalized momentum $\{p_r, p_\phi, p_z\}$

$$\begin{aligned}\mathcal{H} &= \{\dot{q}\}\cdot\{p\} - \mathcal{L} = m\dot{r}^2 + mr^2\dot{\phi}^2 + \frac{e}{2}Br^2\dot{\phi} + m\dot{z}^2 - \frac{m}{2}(\dot{r}^2 + r^2\dot{\phi}^2 + \dot{z}^2) - \frac{e}{2}Br^2\dot{\phi} \\ &= \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\phi}^2 + \dot{z}^2\right) \\ &= \frac{p_r{}^2}{2m} + \frac{\left(p_\phi - \frac{e}{2}Br^2\right)^2}{2mr^2} + \frac{p_z{}^2}{2m}\end{aligned}$$

- Magnetic moment derived from the Hamiltonian, and after solving the equation of motion. (The motion of the particles becomes 2D.)

$$\boldsymbol{\mu} = -\frac{\partial\mathcal{H}}{\partial B}\hat{\mathbf{b}} = \frac{e}{2m}\left(p_\phi - \frac{e}{2}Br^2\right)\hat{\mathbf{b}} = -\frac{e^2B}{2m}r^2$$

- The Hamiltonian and the **partition function** of an ensemble of thermal particles

$$\mathcal{H} = \sum_{i=1}^{n}\mathcal{H}_i,\quad \mathcal{H}_i = \frac{p_{ri}{}^2}{2m_i} + \frac{\left(p_{\phi i} - \frac{e_i}{2}Br_i^2\right)^2}{2m_ir_i^2} + \frac{p_{zi}{}^2}{2m_i},\quad \boxed{Z = \int e^{-\frac{\mathcal{H}}{k_BT}}d\Omega = \prod_{j=1}^{n}Z_j,\quad Z_j = \int e^{-\frac{\mathcal{H}_j}{k_BT}}d\Omega_j}$$

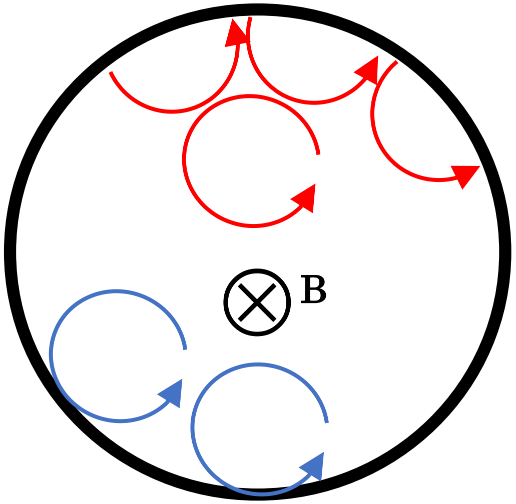
- The magnetic moment of an ensemble of thermal particles (Plasma is a non-linear diamagnetic fluid.)

$$\mathbf{M} = \frac{k_BT}{VZ}\left(\frac{\partial Z}{\partial B}\hat{\mathbf{b}}\right) = -\frac{Nk_BT}{2B}\mathbf{b} = -\frac{P_\perp}{B}\mathbf{b}$$

Bohr-van Leeuwen Paradox

- Furthermore, Bohr stated that
  - even if the gyration motion of the charged particles generated magnetic moments, all the internal magnetic moments would be canceled out by the drift currents due to reflections at the boundary, shown in red.
  - However, he did not consider currents flowing tangentially to the boundary, shown in blue.
  - The magnetization current due to the magnetic moment (∇x M) consists of a ∇B component and a ∇P component. The ∇B component is canceled by the ∇B drift currents, while the ∇P component remains. The system as a whole has a magnetic moment.

(There are charged particles inside but not outside. B is weaker inside than outside.)



$$\nabla \times \mathbf{M} = \nabla \times \left(-\frac{P_\perp}{B} \hat{\mathbf{b}} \right) = \left(-\frac{\nabla P_\perp}{B} + \frac{P_\perp}{B} \frac{\nabla B}{B} \right) \times \hat{\mathbf{b}} - \frac{P_\perp}{B} \left(\nabla \times \hat{\mathbf{b}} \right)$$

Influence of the Bohr-Van Leeuwen paradox

- The statistical-mechanical treatment of matter in classical physics states that magnetic moments do not exist due to the 'Bohr-van Leeuwen theorem' which has been believed for more than 100 years. Therefore, the magnetism of matter can only be explained by quantum mechanics, and MHD does not include magnetic moments.
  - Feynman[1964]"The Feynman Lecture on Physics" (§ 34-6) "Classical physics gives neither diamagnetism nor paramagnetism"
  - Fowler [1936]"Statistical Mechanics"
  - Kittel [1958]"Elementary Statistical Physics" , etc.
- References

1. Thesis work by Bohr：Nielsen[1972]: "Niels Bohr Collected Works, Vol. 1 Early Work (1905-1911)" edited by. J. R. Nielsen, North-Holland Publishing Company, 1972.
2. Thesis work by van Leeuwen[1921]: "Problèmes de la théorie électronique du magnétisme" by van Leeuwen, H. -J., J. Phys. Radium, 2 (12) pp. 361-377, 1921.
3. Introduction of 1. and 2. by van Vleck[1932]: "The theory of electric and magnetic susceptibilities"by J. H. van Vleck, Oxford press, 1932.

Treatment of magnetic moment in MHD

1. Parker[2007] p.54 last paragraph
   - The essential point for our conversation of electric and magnetic fields in the cosmos is that the hot gases that are everywhere in the cosmos have very little electric polarizability and very little magnetic susceptibility, so **D**=**E** and **H**=**B** to good approximation.
2. Cowling[1976] p.3 line 4
   - The material is assumed nonmagnetic: thus its permeability μ has the value $4\pi \times 10^{-7}$ henry/m characteristic of free space.
3. Priest[1982] p.73 after Eq(2.4)
   - where constitutive relations $H = B/\mu$, $D = \varepsilon E$ have been used to eliminate the magnetic field (H) and electric displacement (D). (For solar plasma, μ and ε are invariably approximated by their vacuum values, μ_0 and ε_0, respectively)
4. Landau & Lifshitz[1984] P.225 Chapter VIII, § 65 3rd para
   - The magnetic permeability of the media considered in magnetohydrodynamics differs only slightly from unity, and the difference is unimportant as regards the phenomena under discussion. We shall therefore take $\mu = 1$ through the present chapter.
5. Chen[1984] p. 55 line 1, p.56 line 8
   - In a plasma, the ions and electrons comprising the plasma are the equivalent of the “bound” charges and currents. Since these charges move in a complicated way, it is impractical to try to lump their effects into two constants ε and μ. Consequently, in plasma physics, one generally works with the vacuum equations [3-1] – [3-4], in which σ and**j** include all the charges and currents, both external and internal.
   - In a plasma with a magnetic field, each particles has a magnetic moment μ_α, and the quantity **M** is the sum of all theseμ_α’s in 1 m^3. But we now have

$$\mu_\alpha = \frac{mv_{\perp\alpha}^2}{B} \propto \frac{1}{B} \quad M \propto \frac{1}{B}$$

The relation between **M** and **H** (or **B**) is no longer linear, and we cannot write $\mathbf{B} = \mu_m \mathbf{H}$ with $\mu_m$constant. It is therefore not useful to consider a plasma as a magnetic medium.

The constitutive equation of magnetic field and Maxwell equations for non-linear medium

- The constitutive equation is,

$$\mathbf{B}/\mu_0 = \mathbf{H} + \mathbf{M} = \mathbf{H} - \left(P_\perp / B\right)\mathbf{b}, \quad \mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}_e$$

  - Plasma is a non-linear diamagnetic medium.

- Maxwell equations in a medium are,

$$\nabla \times \mathbf{H} - \partial_t \mathbf{D} = \mathbf{J}_{free}, \quad \nabla \cdot \mathbf{D} = \rho_{free}, \quad \nabla \times \mathbf{E} + \partial_t \mathbf{B} = 0, \quad \nabla \cdot \mathbf{B} = 0$$

  - To accommodate non-linearity, bound current and charge are introduced.

$$\nabla \times \mathbf{B}/\mu_0 - \varepsilon_0 \partial_t \mathbf{E} = \mathbf{J}_{total}, \quad \varepsilon_0 \nabla \cdot \mathbf{E} = \rho_{total}, \quad \nabla \times \mathbf{E} + \partial_t \mathbf{B} = 0, \quad \nabla \cdot \mathbf{B} = 0$$

$$\text{where} \quad \mathbf{J}_{total} = \mathbf{J}_{free} + \nabla \times \mathbf{M} + \partial_t \mathbf{P}_e \quad \text{and} \quad \rho_{total} = \rho_{free} - \nabla \cdot \mathbf{P}_e$$

- The assumption of low beta in MHD is equivalent to neglecting the magnetic moment.

$$\frac{-M}{B/\mu_0} = \left(\frac{1}{2}\right) \frac{P_\perp}{B^2/(2\mu_0)} = \beta/2$$

Magnetic Kelvin force

- Electromagnetic momentum conservation law is:

$$\nabla\cdot\left(\varepsilon_0\mathbf{EE}+\frac{\mathbf{BB}}{\mu_0}\right)-\nabla\left(\frac{\varepsilon_0}{2}E^2+\frac{B^2}{2\mu_0}\right)-\frac{\partial}{\partial t}\left(\varepsilon_0\mathbf{E}\times\mathbf{B}\right)=\rho_{total}\mathbf{E}+\mathbf{J}_{total}\times\mathbf{B}$$

- The Lorentz force acting on the magnetization current is:

$$\left(\nabla\times\mathbf{M}\right)\times\mathbf{B}=\nabla\cdot\left(\mathbf{BM}\right)-\nabla\left(\mathbf{B}\cdot\mathbf{M}\right)+\mathbf{M}\cdot\left(\nabla\mathbf{B}\right)^T,\quad\nabla\cdot\mathbf{B}=0$$

- The magnetic Kelvin force is:

$$\mathbf{F}_K=\mathbf{M}\cdot\left(\nabla\mathbf{B}\right)^T=\left(\mathbf{M}\cdot\frac{\partial\mathbf{B}}{\partial x},\mathbf{M}\cdot\frac{\partial\mathbf{B}}{\partial y},\mathbf{M}\cdot\frac{\partial\mathbf{B}}{\partial z}\right)$$

- The Kelvin force along the field acting on the thermal plasma:

$$\mathbf{M}=-\frac{P_\perp}{B}\hat{\mathbf{b}},\quad \mathrm{F}_{K\parallel}=-\frac{P_\perp}{B}\frac{\partial B}{\partial s}$$

- Equation of motion of thermal plasma along the field:

$$\rho_m\left\{\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial s}\right\}=-\frac{\partial P_\parallel}{\partial s}-\frac{P_\perp}{B}\frac{\partial B}{\partial s}-\rho_m g_\parallel-\alpha\rho_m u$$

Application to plasma in the solar atmosphere

- Equation of motion of plasma in the solar atmosphere along the field:

$$\rho_m \left\{ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial s} \right\} = -\frac{\partial P_\parallel}{\partial s} - \frac{P_\perp}{B} \frac{\partial B}{\partial s} - \rho_m g_0 \cos(\theta) - \alpha \rho_m u$$

  - Assumption: plasma consists of protons (N/2) and electrons (N/2)

$$\rho_m = (1/2) N m_p, \quad P_\parallel = N k_B T, \quad P_\perp = (1/2) N k_B T$$

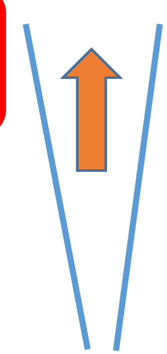
α : collision rate with background neutrals

- Constant velocity upward flow without pressure gradient force

$$u = \frac{g_0}{\alpha} \left(\frac{T}{T_c} - \cos(\theta) \right), \quad \frac{1}{\ell_B} = -\frac{1}{B} \frac{\partial B}{\partial s}, \quad \boxed{T_c = \frac{m_p g_0}{k_B} \ell_B \sim 1.7 \times 10^6\,\mathrm{K}}$$

$$m_p = 1.67 \times 10^{-27}\,\mathrm{kg}, \quad k_B = 1.38 \times 10^{-23}\,\mathrm{JK}^{-1}, \quad g_0 = 2.74 \times 10^2\,\mathrm{ms}^{-2}$$

$$\ell_B \sim 100\,\mathrm{Mm}/2 = 5 \times 10^7\,\mathrm{m} \text{ (half of the active region size)}$$



- Upward flow condition is $T > T_c \cos(\theta)$

Application to plasma in the solar atmosphere

- Steady hot plasma outflow is due to the Kelvin force.
- Temperature dependent plasma up- or down-flow is due to the Kelvin force.
- Loop-top plasma concentration
  - In the closed magnetic loop, hot plasma is concentrated at the top.

- Coronal heating:
  - Hot plasmas are lifted into the corona by the Kelvin force (corona is ≧ 1MK).
  - Coronal plasmas are supplied from below the photosphere through magnetic flux tubes.
- Solar wind acceleration:
  - If collisional friction is negligible, plasma will accelerate upward due to the $u\partial u/\partial s$ term.